International Journal of Theoretical Physics, Vol. 14, No. 5 (1975), pp. 319–326

Some Type III Solutions of the Einstein–Maxwell Equations

S. HACYAN

Instituto de Astronomía, Universidad Nacional Autónoma de México, México 20, D.F.

J. F. PLEBAŇSKI*

Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, México 14, D.F.

Received: 3 April 1975

Abstract

A Petrov type III metric with nontwisting, degenerate Debever–Penrose direction is studied. This metric is, in general, a solution of the Einstein–Maxwell equations. Two particular cases are investigated in some detail. It is shown that the metric contains type N, conformally flat and flat metrics as special subcases. Among these subcases, we find the metric of plane gravitational waves and the Bertotti–Robinson solution.

1. *Introduction*

This work investigates a class of metrics of the form

$$ds^2 = 2e^{2\alpha} dz d\bar{z} + 2du(dv - \beta du) \tag{1.1}$$

(signature: +++–), where u and v are two real coordinates, z is a complex coordinate (bars denote complex conjugation), and α and β are two real functions. We set $G = 1 = c$.

Metrics of the type (1.1) contain some interesting solutions of Einstein–Maxwell equations. These solutions are of Petrov type III, and contract to the type N and then to the conformally flat space. In particular, we obtain a solution which represents a plane gravitational wave in the presence of a static electromagnetic field.

2. *General Results*

Some metrics which describe gravitational waves together with electromagnetic radiation are already known (Zakharov, 1973). In this work we consider

* On leave of absence from University of Warsaw, Warsaw, Poland.

the electromagnetic field of a static type, with the field tensor algebraically general; that is, it has two different real null eigenvectors. Now, suppose we orient one of these eigenvectors along a Debever-Penrose (DP) vector (Debney et al., 1969); then, the following theorem can be proved (Plebañski, 1974): if e^3 is a real eigenvector of the (algebraically general) electromagnetic field and if it is also an at least triple Debever-Penrose vector, then $\Gamma_{42} = 0$, that is, e^3 is geodesic, shearless, and without expansion or rotation. [Notations and conventions of Debney et al. (1969) are used throughout this paper.] The proof of this theorem follows from Bianchi identities and Maxwell equations and is similar to the Goldberg-Sachs (1969) theorem. Kundt (1961), working with $\Gamma_{42} = 0$, studied plane-fronted gravitational waves propagating in the direction e^3.

We will now consider solutions of the Einstein-Maxwell equations endowed with the property that one real eigenvector of the algebraically general electromagnetic field is parallel to an (at least) triple Debever-Penrose vector.

It turns out that the simple metric (1.1) provides solutions of this type. The natural null-tetrad associated to metric (1.1) is

$$\begin{aligned} e^1 &= e^{\alpha} dz \\ e^2 &= e^{\alpha} d\bar{z} \\ e^3 &= du \\ e^4 &= dv - \beta du \end{aligned} \tag{2.1}$$

and we choose e^3 as the distinguished null vector. Obviously

$$ds^2 = 2e^1e^2 + 2e^3e^4 \tag{2.2}$$

The inverse of this tetrad, defined by $e^a = e_a{}^{\mu}\,\partial_{\mu}$, is

$$\begin{aligned} \partial_1 &= e^{-\alpha}\partial_z \\ \partial_2 &= e^{-\alpha}\partial_{\bar{z}} \\ \partial_3 &= \partial_u + \beta\partial_v \\ \partial_4 &= \partial_v \end{aligned}$$

Now, from the first structure equations, $de^a = e^n \wedge \Gamma^a n$, and $\Gamma_{ab} = -\Gamma_{ba}$, one easily finds the independent connection forms Γ_{ab}:

$$\begin{aligned} \Gamma_{12} + \Gamma_{34} &= -\alpha_{,z} dz + \alpha_{,\bar{z}} d\bar{z} - \beta_{,v} du \\ \Gamma_{31} &= -e^{\alpha}\alpha_{,u} d\bar{z} - e^{-\alpha}\beta_{,z} du \\ \Gamma_{42} &= 0 \end{aligned} \tag{2.4}$$

(colons denote partial derivatives). Condition $\Gamma_{42} = 0$ is a consequence of the theorem stated above, but it applies to the studied metric with the additional assumption that

$$\alpha_{,v} = 0 \tag{2.5}$$

Knowing Γ_{ab}, we can now compute the tetrad components of the curvature tensor from Cartan's structure formulas:

$$d\Gamma^a{}_b + \Gamma^a{}_n \wedge \Gamma^n{}_b = \tfrac{1}{2}R^a{}_{bcd}e^c \wedge e^d \tag{2.6}$$

where $R^a{}_{bcd}$ are the tetradial components of the Riemann tensor. The final result is the following: if $C^{(a)}$ $(a = 1, \ldots, 5)$ are the usual five functions which describe conformal curvature and R_{ab} are the tetrad components of the Ricci tensor, one finds

$$C^{(5)} = C^{(4)} = C^{(3)} = 0 \tag{2.7a}$$

$$C^{(2)} = e^{-\alpha}\partial_z(\beta_{,v} - \alpha_{,u}) \tag{2.7b}$$

$$C^{(1)} = 2\partial_z(e^{-2\alpha}\beta_{,z}) \tag{2.7c}$$

and the only nonzero components of R_{ab} are

$$R_{12} = -\beta_{,vv} = 2e^{-2\alpha}\alpha_{,z\bar{z}} = -R_{34} \tag{2.8a}$$

$$R_{31} = e^{-\alpha}\partial_z(\beta_{,v} + \alpha_{,u}) = \overline{R_{32}} \tag{2.8b}$$

$$R_{33} = 2(\alpha_{,uu} + (\alpha_{,u})^2 - e^{-2\alpha}\beta_{,z\bar{z}} - \beta_{,v}\alpha_{,u}) \tag{2.8c}$$

Moreover, the Ricci tensor turns out to be traceless:

$$R = 0 \tag{2.9}$$

[It also follows, directly from structure formulas (2.6), that, if $\Gamma_{42} = 0$, the cosmological constant must vanish if one of the DP vectors is (at least) triply degenerated.]

Notice that eq. (2.8a) and condition (2.5) imply that β is a second-order polynomial in v. Thus, we set

$$\beta = A(z, \bar{z}, u)v^2 + B(z, \bar{z}, u)v + C(z, \bar{z}, u) \tag{2.10}$$

Now, the Einstein-Maxwell equations, written in tensorial notation, are

$$f^{\mu\nu}{}_{;\nu} = 0 = \check{f}^{\mu\nu}{}_{;\nu} \tag{2.11}$$

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8_\pi E_{\mu\nu} \tag{2.12}$$

where $f_{\mu\nu}$ is the electromagnectic field tensor,

$$4\pi E_{\mu\nu} = -f_{\mu\rho}f_\nu{}^\rho + \tfrac{1}{4}g_{\mu\nu}f_{\rho\sigma}f^{\rho\sigma} \tag{2.13}$$

and the duality operation is defined by

$$\check{f}^{\mu\nu} := (i/2\sqrt{-g})\epsilon^{\mu\nu\rho\sigma}f_{\rho\sigma} \quad (g := \det \|g_{\mu\nu}\|) \tag{2.14}$$

The complex invariant of the electromagnetic field is

$$\mathscr{F} = \tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} + \tfrac{1}{4}f_{\mu\nu}\check{f}^{\mu\nu} \tag{2.15}$$

and we must have $F \neq 0$ if the field is algebraically general. Finally, the complex two-form of the electromagnetic field is

$$\omega = \tfrac{1}{2}(f_{\mu\nu} + \check{f}_{\mu\nu}) dx^\mu \wedge dx^\nu \tag{2.16}$$

and, if Maxwell equations without currents are satisfied, it must be closed, i.e.,

$$d\omega = 0 \tag{2.17}$$

The tetradial components of the energy-momentum tensor, E_{ab}, satisfy the algebraic relation

$$E_{12}E_{33} - E_{31}E_{32} = 0 \tag{2.18}$$

If the Einstein equations (2.12) are satisfied, this last relation must also hold for R_{ab}:

$$R_{12}R_{33} - R_{31}R_{32} = 0 \tag{2.19}$$

This relation contains terms in v^2, v, and without v. This gives three relations:

$$AA_{,z\bar{z}} - A_{,z}A_{,\bar{z}} = 0 \tag{2.20a}$$

$$\begin{aligned} 2A[B_{,z\bar{z}} + 2Ae^{2\alpha}\alpha_{,u}] - A_{,z}\partial_{\bar{z}}(B + \alpha_{,u}) \\ - A_{,\bar{z}}\partial_z(B + \alpha_{,u}) = 0 \end{aligned} \tag{2.20b}$$

$$\begin{aligned} A[\alpha_{,uu} + (\alpha_{,u})^2 - e^{-2\alpha}C_{,z\bar{z}} - B\alpha_{,u}] \\ + \tfrac{1}{4}e^{-2\alpha}\partial_z(B + \alpha_{,u})\partial_{\bar{z}}(B + \alpha_{,u}) = 0 \end{aligned} \tag{2.20c}$$

From (2.20a) it follows that

$$A = |Q(u, z)|^2 \tag{2.21}$$

and, using (2.8a),

$$\alpha_{,z\bar{z}} + |Q|^2 e^{2\alpha} = 0 \tag{2.22}$$

This is known as Liouville's equation (Goursat, 1923) and has a simple solution:

$$e^{2\alpha} = |Q|^{-2} \frac{|F_{,z}|^2}{2(1 + \frac{1}{2}|F|^2)^2} \tag{2.23}$$

where $F = F(u, z)$ is an arbitrary function.

To find a general solution of Eqs. (2.20) in a plausibly closed form is rather difficult. However, some interesting particular solutions can be determined; we will present two simple and plausible subcases in the following sections.

3. *First Particular Solution*

Let

$$\begin{aligned} A &= |Q|^2 = \tfrac{1}{2}|F|^2 \\ B &= 0 \\ C &= -(\phi + \bar{\phi}) \end{aligned} \tag{3.1}$$

where $F = F(z)$ is an holomorphic function of z, with derivative F', and $\phi = \phi(u, z)$. According to (2.23), we have

$$e^{2\alpha} = 1/(1 + \tfrac{1}{2}|F|^2)^2 \tag{3.2}$$

and it is easy to see that (3.1) and (3.2) are particular solutions of (2.20). The metric has the form

$$ds^2 = 2dzd\bar{z}/(1 + \tfrac{1}{2}|F|^2) + 2du[dv + (\phi + \bar{\phi} - \tfrac{1}{2}v^2|F'|^2)\,du] \tag{3.3}$$

the conformal curvature functions are

$$\begin{aligned} C^{(1)} &= 2\partial_z[(1 + \tfrac{1}{2}|F|^2)^2(\tfrac{1}{2}F''\overline{F'}v^2 - \phi_{,z})] \\ C^{(2)} &= (1 + \tfrac{1}{2}|F|^2)F''\overline{F'}v \end{aligned} \tag{3.4}$$

and the electromagnetic two-form is

$$\omega = e^{i\psi} d\left(\frac{F}{1 + \tfrac{1}{2}|F|^2}\, d\bar{z} - F'v\,du\right) \tag{3.5}$$

Here, ψ represents the arbitrary phase of the duality rotations with precision up to which $E_{\mu\nu}$ determines $f_{\mu\nu}$. However, the Maxwell equations imply $d\omega = 0$, and this condition is fulfilled if and only if

$$\psi = \text{const} \tag{3.6}$$

In the following, we set $\psi = 0$ without losing generality (i.e., we absorb it redefining $F \to Fe^{-i\psi}$). Finally, the electromagnetic field invariant is simply given by

$$\mathscr{F} = -\tfrac{1}{2}(F')^2 \tag{3.7}$$

The functions F and ϕ are in general arbitrary, but some special subcases are of interest:

Subcase (a): $F'' \neq 0$. The metric is of type III, with an algebraically general electromagnetic field.

Subcase (b): $\phi_{,zz} \neq 0$ $F = const$. The metric has the form

$$ds^2 = 2dzd\bar{z} + 2du\,[dv + (\phi + \bar{\phi})du] \tag{3.8}$$

and all $C^{(a)}$'s are zero, except

$$C^{(1)} = -2\phi_{,zz} \tag{3.9}$$

Thus, metric (3.8) is of type N. It is the well-known metric of a plane gravitational wave in vacuum (Robinson, 1956).

Subcase (c): F' = const. $\neq 0$ and $\phi_{,z} \neq 0$. The electromagnetic field invariant, $\mathscr{F}$ is here constant and all the $C^{(a)}$'s are zero, except

$$C^{(1)} = -2v^2 \partial_z [\phi_{,z}(1 + \tfrac{1}{2}F\bar{F})^2] \tag{3.10}$$

Thus, in this particular case, the metric is of type N, while the nontrivial electromagnetic field is algebraically general.

Subcase (d): F' = const $\neq 0$, $\phi_{,z} = 0$. The metric is conformally flat and the electromagnetic field is homogeneous: $f_{\alpha\beta;\gamma} = 0$. This subcase is the Bertotti-Robinson solution (Bertotti, 1959).

Subcase (e): Flat space limit. Gaussian units can be restored in all the formulas given above by simply making the substitution

$$\begin{aligned} F &\to (\sqrt{G}/c^2)F \\ R_{ab} &\to (G/c^4)R_{ab} \end{aligned} \tag{3.11}$$

Transition to flat space is achieved by taking the limit $G \to 0$ and setting $\phi_{,zz} = 0$ [see (3.4)]. The electromagnetic two-form reduces to

$$\omega = d(F\,d\bar{z} - F'v\,du) \tag{3.12}$$

and the metric takes the form

$$ds^2 = 2dz\,d\bar{z} + 2du[dv + (\phi + \bar{\phi})du] \tag{3.13}$$

with the condition that

$$\phi = a(u)z + b(u) \tag{3.14}$$

Metric (3.13) exhibits "spurious" gravitational waves; that is, they can be removed by a proper change of coordinates. For instance, if a = const and $b = 0$ in (3.14), metric (3.13) can be transformed to

$$\begin{aligned} ds^2 &= 2d(z - \tfrac{1}{2}au^2)d(\bar{z} - \tfrac{1}{2}\bar{a}u^2) \\ &\quad + 2du\,d[-\tfrac{1}{3}a\bar{a}\,u^3 + v + (a\bar{z} + \bar{a}z)u] \end{aligned} \tag{3.15}$$

which is obviously flat. Thus, subcase (e) is a special solution of Maxwell's equation in flat space, given in a somewhat unusual coordinate system.

In conclusion, subcase (b) suggests that the metric (3.3), in its most general form, is a generalization of a metric describing plane gravitational waves when a static electromagnetic field is present. This electromagnetic field is not homogeneous except in subcase (d). Finally, we saw that e^3 is at least a triple Debever-Penrose vector and at the same time one of the real eigenvectors of the algebraically general electromagnetic field. If we define

$$e^{4'} = e^4 + \rho e^1 + \bar{\rho}e^2 - \rho\bar{\rho}e^3 \tag{3.16}$$

then $e^{4'}$ is the second eigenvector of the electromagnetic field if

$$\rho = \tfrac{1}{2}(1 + \tfrac{1}{2}|F|^2)(F''/F')v \tag{3.17}$$

Similarly, if we set

$$\rho = -\tfrac{1}{4}\, C^{(1)}/C^{(2)} \tag{3.18}$$

$e^{4'}$ amounts to the single Debever-Penrose vector of the type III. It can be shown that in neither case is $e^{4'}$ geodesic.

4. *Second Particular Solution*

Let

$$Q = \text{const} \neq 0 \to A = |Q|^2 = \text{const} \neq 0 \tag{4.1a}$$

$$B = \alpha_{,u} + H(u, z) + \bar{H}(u, \bar{z}) \tag{4.1b}$$

where H is an arbitrary function of u and z, and $C(u, z, \bar{z})$ is any integral of the Poisson equation:

$$C_{,z\bar{z}} = e^{2\alpha}[\alpha_{,uu} - \alpha_{,u}(H + \bar{H})] + (1/4A)|\partial_z(2\alpha_{,u} + H)|^2 \tag{4.1c}$$

Formulas (4.1) define another particular solution of Eqs. (2.20). The metric takes the form

$$ds^2 = |Q|^{-2} \frac{|F_{,z}|^2 dz d\bar{z}}{(1 + \frac{1}{2}|F|^2)^2} + 2du(dv - \beta du) \tag{4.2}$$

where $F = F(u, z)$ is an arbitrary function and β is defined by (2.10) and (4.1). The conformal curvature functions, $C^{(a)}$, which are different from zero are

$$\begin{aligned} C^{(1)} &= 2\partial_z\{e^{-2\alpha}[(\alpha_{,uz} + H_{,z})v + C_{,z}]\} \\ C^{(2)} &= e^{-\alpha}H_{,z} \end{aligned} \tag{4.3}$$

and the form of the electromagnetic field is

$$\begin{aligned} \omega &= (e^{i\psi}/\sqrt{2}\bar{Q})d[2\alpha_{,z}dz - (2Av + H)du] \\ F &= -A \end{aligned} \tag{4.4}$$

[The same comment as after (3.5) applies to ψ in this last equation; in this case, we can absorb ψ by redefining $Q \to Qe^{-2i\psi}$). It can be shown that this electromagnetic field is homogeneous along direction e^4, i.e.,

$$e^{4\mu} f_{\alpha\beta;\mu} = 0 \tag{4.5}$$

It is clear from (4.3) that the metric is of type III if $H_{,z} \neq 0$ and, correspondingly, $C^{(1)} \neq 0$ or $C^{(1)} = 0$). The transition to flat space can be achieved by setting, in (4.2),

$$\begin{aligned} |Q|^2 &\to (G/c^4)|Q|^2, \quad A \to (G/c^4)A \\ \beta &\to (G/c^4)Av^2, \quad F \to (\sqrt{G}/c^2)F \end{aligned} \tag{4.6}$$

and then taking the limit $G \to 0$. One obtains a solution of Maxwell equations in flat space. The electromagnetic form is

$$\omega = \sqrt{2} Q e^{i\psi} d(\tfrac{1}{2} z \, d\bar{z} - v \, du) \tag{4.7}$$

As in the previous case, we can define $e^{4'}$ as in (3.16); then, if

$$\rho = (1/8\sqrt{2}\bar{Q}) e^{-\alpha} \partial_z (2\alpha_u + H) \tag{4.8}$$

$e^{4'}$ is the second eigenvector of the electromagnetic field, and, if ρ is as in (3.18) but with $C^{(1)}$ and $C^{(2)}$ given by (4.3), $e^{4'}$ is the single Debever-Penrose vector. Again, in neither case is $e^{4'}$ geodesic.

5. *Final Remarks*

We can now sum up the results obtained. Metric (1.1) is of type III and contains type N, conformally flat, and flat metrics as special subcases. This can be interpreted as a contraction scheme along the line of null gravitational invariants in Penrose's diagram. Schematically we have

$$[\text{flat}] \leftarrow [-] \leftarrow [4] \leftarrow [3\text{-}1]$$

in the notation of Penrose (1960).

It is clear from the study of the two particular solutions given above that, in general, the single Debever-Penrose vector is not parallel to an eigenvector of the electromagnetic field. This, in fact is a general feature: it can be shown, after some lengthy algebra, that no type III solution of Einstein-Maxwell equations exists such that *both* Debever-Penrose vectors are parallel to the two eigenvectors of the electromagnetic field.

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